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Tail probability of random variable and Laplace transform

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We investigate the exponential decay of the tail probability $P(X > x)$ of a continuous type random variable X. Let $\phi(s)$ be the Laplace–Stieltjes transform of the probability distribution function $F(x) = P(X \le x)$ of X, and σ_0 be the abscissa of convergence of $\phi(s)$. We will prove that if $-\infty < \sigma_0 < 0$ and the singularities of $\phi(s)$ on the axis of convergence are only a finite number of poles, then the tail probability decays exponentially. For the proof of our theorem, Ikehara's Tauberian theorem will be extended and applied.

Keywords: Tail probability of random variable; Exponential decay; Laplace transform; Tauberian theorem

Mathematics Subject Classifications (2000): Primary 44A10; 60A99

1. Introduction

In various engineering problems, it is often important to evaluate the tail probability of a random variable. For instance, the tail probability of the waiting time in a queue is used for the design of buffer size and call admission control in communication networks. Further, the error probability of an error correcting code is also evaluated by the tail probability. In large deviations theory, one considers the probability that the average of a sequence of random variables exceeds some value and discusses the exponential decay of the tail probability when the number of random variables becomes infinite. In this article, we consider how the tail probability $P(X > x)$ of one random variable X decays as x tends to infinity.

We say that the tail probability $P(X > x)$ *decays exponentially* if

$$P(X > x) \simeq e^{\sigma_0 x}, \quad \sigma_0 < 0,$$

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or, more precisely,

$$\lim_{x\to\infty}\frac{1}{x}\log P(X > x) = \sigma_0, \quad \sigma_0 < 0. \tag{1}$$

σ_0 is called the *exponential decay rate*. We give a condition on which the tail probability decays exponentially. We consider the Laplace–Stieltjes transform $\phi(s)$ of the probability distribution function $F(x) = P(X \le x)$, and investigate the decay of the tail probability $P(X > x)$ based on analytic properties of $\phi(s)$.

Even in a case that we cannot obtain the explicit form of $P(X > x)$, we sometimes can obtain $\phi(s)$ explicitly. For example, in queueing theory, we may calculate the Laplace–Stieltjes transform $\phi(s)$ of the waiting time by the Pollaczek–Khinchin formula. In such a case, if $\phi(s)$ is inversely transformed easily, we could obtain the tail probability; however, inversion is difficult in general. Since our purpose is to study the exponential decay of the tail probability, it is not exactly essential to know the explicit form of the tail probability itself. Rather, it is enough to focus only on the properties of $\phi(s)$ for investigating the exponential decay.

We can readily see that if $\phi(s)$ is a rational function, then the tail probability $P(X > x)$ decays exponentially since $\phi(s)$ can be expanded by partial fractions and each fraction has the inverse transform which decays exponentially. In the case of a rational function, the exponential decay and the decay rate are determined by the poles with the maximum real part. The poles with the maximum real part lie on the axis of convergence of $\phi(s)$, and other poles do not contribute to the exponential decay. From these facts, we can expect that even when $\phi(s)$ is not rational the poles on the axis of convergence determine the exponential decay of $P(X > x)$.

The author discussed in [4] the exponential decay of the tail probability $P(X > x)$ of a discrete type random variable X. We considered the probability generating function

$$\pi(z) = \sum_{n=0}^{\infty} P(X = n)z^n$$

of X and investigated the exponential decay of $P(X > x)$ by noticing complex analytic properties of $\pi(z)$. In particular, we elucidated the relation between the exponential decay of $P(X > x)$ and the radius of convergence and the number of poles of $\pi(z)$ on the circle of convergence.

Denote by r the radius of convergence of $\pi(z)$, then Cauchy–Hadamard's formula [3] implies

$$\limsup_{n\to\infty}\frac{1}{n}\log P(X = n) = -\log r.$$

In addition, if $1 < r < \infty$, then

$$\limsup_{n\to\infty}\frac{1}{n}\log P(X > n) = -\log r. \tag{2}$$

In [4], we discussed conditions that guarantee the existence of the limit (instead of the lim sup) of the left hand side of (2), and showed the following:

(1) If $1 < r < \infty$ and the singularities of $\pi(z)$ on the circle of convergence $|z| = r$ are only a finite number of poles, then $\lim_{n\to\infty} n^{-1} \log P(X > n)$ exists and is equal to $-\log r$.
(2) We gave an example of a discrete type random variable X with $\limsup_{n\to\infty} n^{-1} \log P(X > n) \neq \liminf_{n\to\infty} n^{-1} \log P(X > n)$.
(3) We gave an example of M/G/1 type Markov chain whose stationary distribution does not have an exponentially decaying tail.

In this article, we will extend the results in [4] to continuous type random variables. While we considered the probability generating function in the case of a discrete type random variable, we focus on the Laplace–Stieltjes transform of the probability distribution function $F(x)$ if X is of continuous type. Let $\phi(s)$ be the Laplace–Stieltjes transform of $F(x)$ and σ_0 be the abscissa of convergence of $\phi(s)$. The vertical line $\Re s = \sigma_0$ is called the axis of convergence of $\phi(s)$, where $\Re s$ represents the real part of s. The main theorem of this paper is that if $-\infty < \sigma_0 < 0$ and the singularities of $\phi(s)$ on the axis of convergence are only a finite number of poles, then

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) = \sigma_0. \tag{3}$$

The biggest difference between the case of discrete type random variables and that of continuous ones is the topological property of the domain of convergence of power series and Laplace–Stieltjes transform. In the power series the circle of convergence $|z| = r$ is a compact set, while the axis of convergence $\Re s = \sigma_0$ is not compact. Due to this difference, although the theorems for both types of random variables formally look similar, the proof for the continuous type random variables is much more complicated than that for discrete ones.

This article is organized as follows. In section 2, the definition of the Laplace–Stieltjes transform and its abscissa of convergence are given and some lemmas are presented. In section 3, we show an example of a random variable X whose tail probability $P(X > x)$ does not decay exponentially. In section 4, Tauberian theorems are introduced for the purpose of proving our main theorem. In section 5, some preliminary lemmas are given for the proof of the main theorem. Ikehara's Tauberian theorem is extended to the case that the Laplace transfrom has a finite number of poles on the axis of convergence. In section 6, the main theorem is proved by using the lemmas proved in section 5. Further, we show that for the example $F^*(x)$ in section 3 all the points on the axis of convergence are singularities of the Laplace–Stieltjes transform of $F^*(x)$. In section 7, we conclude with the results.

2. Laplace–Stieltjes transform of probability distribution function and its abscissa of convergence

Let X be a non-negative continuous type random variable with probability distribution function

$$F(x) = P(X \leq x).$$

Define the Laplace–Stieltjes transform of $F(x)$ as

$$\phi(s) = \int_0^\infty e^{-sx} dF(x), \ \ s = \sigma + i\tau. \tag{4}$$

The *abscissa of convergence* of $\phi(s)$ is defined as the number σ_0 such that the integral (4) converges for $\Re s > \sigma_0$ and diverges for $\Re s < \sigma_0$. The vertical line $\Re s = \sigma_0$ is called the *axis of convergence* of $\phi(s)$. Since $\phi(0) = 1 < \infty$, we have $\sigma_0 \leq 0$.

Now, we consider the *tail probability*

$$\tilde{F}(x) = P(X > x)$$

of the random variable X. We have $F(x) + \tilde{F}(x) = 1$.

Let σ_0 denote the abscissa of convergence of (4). We can easily obtain

LEMMA 1 *The abscissa of convergence of the Laplace–Stieltjes transform*

$$\int_0^\infty e^{-sx} d\tilde{F}(x), \ \ s = \sigma + i\tau$$

of $\tilde{F}(x)$ *is equal to* σ_0.

Next, we have

LEMMA 2 *If* $\sigma_0 < 0$, *then for* s *with* $\Re s > \sigma_0$,

$$\int_0^\infty e^{-sx} dF(x) = 1 - s \int_0^\infty e^{-sx} \tilde{F}(x) dx.$$

Proof See Widder [5], p. 41, Theorem 2.3b. ■

From Lemma 2, we have

LEMMA 3 *If* $\sigma_0 < 0$, *then the abscissa of convergence of the Laplace transform*

$$\psi(s) = \int_0^\infty e^{-sx} \tilde{F}(x) dx$$

of $\tilde{F}(x)$ *is equal to* σ_0.

From Lemma 1 and Widder [5], p. 44, Theorem 2.4e (see Appendix), we have the next lemma.

LEMMA 4 *If* $\sigma_0 < 0$, *then for the tail probability* $\tilde{F}(x)$, *we have*

$$\limsup_{x \to \infty} \frac{1}{x} \log \tilde{F}(x) = \sigma_0. \tag{5}$$

Proof We give a proof different from one in [5]. By Markov's inequality, for any σ with $\sigma_0 < \sigma < 0$, we have

$$\tilde{F}(x) \leq e^{\sigma x} \phi(\sigma). \tag{6}$$

From (6)

$$\frac{1}{x}\log \tilde{F}(x) \leq \sigma + \frac{1}{x}\log \phi(\sigma), \tag{7}$$

hence, by taking lim sup of both sides of (7), we have

$$\limsup_{x\to\infty} \frac{1}{x}\log \tilde{F}(x) \leq \sigma_0, \tag{8}$$

since σ is arbitrary in $\sigma_0 < \sigma < 0$. Suppose that the equality in (8) does not hold, then there should exist σ_1 with $\sigma_1 < \sigma_0$ such that $\tilde{F}(x)$ satisfies

$$\tilde{F}(x) \leq e^{\sigma_1 x}$$

for sufficiently large x. But by Lemma 3 this contradicts σ_0 being the abscissa of convergence. ■

Lemma 4 can be regarded as an analogy of Cauchy–Hadamard's formula on the radius of convergence of power series. In general, the lim sup in (5) is different from lim inf. We will show such an example in the next section.

3. Example of X whose tail probability does not decay exponentially

In [4], an example of a discrete type random variable X is given such that the lim sup in (2) does not coincide with lim inf, i.e., the limit does not exist. In this section, we will show an example of a continuous type random variable such that the lim sup in (5) is not equal to lim inf by modifying the example in [4].

For any $h > 0$, we define a sequence $\{c_n\}_{n=0}^{\infty}$ by the following recursive formula:

$$\begin{cases} c_0 = 0, \\ c_n = c_{n-1} + e^{hc_{n-1}}, \quad n = 1, 2, \ldots. \end{cases} \tag{9}$$

A function $\gamma(x)$, $x \geq 0$ is defined by

$$\gamma(x) = e^{-hc_n}, \ \textit{for}\ c_n \leq x < c_{n+1},\ n = 0, 1, \ldots. \tag{10}$$

$\gamma(x)$ has following properties, which are easily verified.

$$\gamma(x) \text{ is right continuous, piecewise constant, and non-increasing.} \tag{11}$$

$$\int_{c_n}^{c_{n+1}} \gamma(x)dx = e^{-hc_n}(c_{n+1} - c_n) = 1, \quad n = 0, 1, \ldots. \tag{12}$$

$$\int_0^\infty \gamma(x)dx = \infty. \tag{13}$$

$$\text{For any } \eta > 0, \quad \int_0^\infty e^{-\eta x}\gamma(x)dx < \infty. \tag{14}$$

Now, for $\sigma_0 < 0$, putting $F^*(x) = 1 - e^{\sigma_0 x}\gamma(x),\ x \geq 0,$ we find from (11) that $F^*(x)$ is a right continuous and non-decreasing function with $F^*(0) = 0,\ F^*(\infty) = 1$, i.e., $F^*(x)$ is a distribution function. Let us define X^* as a random variable whose probability distribution function is $F^*(x)$. Denoting by $\tilde{F}^*(x)$ the tail probability of X^*, we have

$$\tilde{F}^*(x) = P(X^* > x) = e^{\sigma_0 x}\gamma(x), \quad x \geq 0. \tag{15}$$

Hence, from (13), we have

$$\int_0^\infty e^{-\sigma_0 x}\tilde{F}^*(x)dx = \int_0^\infty \gamma(x)dx = \infty, \tag{16}$$

and for any $\sigma > \sigma_0$, from (14),

$$\int_0^\infty e^{-\sigma x}\tilde{F}^*(x)dx = \int_0^\infty e^{-(\sigma-\sigma_0)x}\gamma(x)dx < \infty. \tag{17}$$

Thus from (16), (17) the abscissa of convergence of the Laplace transform of $\tilde{F}^*(x)$ is σ_0. Therefore, from Lemmas 1, 3, 4, we have

$$\limsup_{x\to\infty} \frac{1}{x}\log \tilde{F}^*(x) = \sigma_0. \tag{18}$$

On the other hand, for $x = c_n$,

$$\begin{aligned} \tilde{F}^*(c_n) &= e^{\sigma_0} c_n \gamma(c_n) \\ &= e^{(\sigma_0 - h)} c_n, \end{aligned}$$

and hence

$$\begin{aligned} \liminf_{x\to\infty} \frac{1}{x}\log \tilde{F}^*(x) &\leq \liminf_{n\to\infty} \frac{1}{c_n}\log \tilde{F}^*(c_n) \\ &= \sigma_0 - h \\ &< \sigma_0. \end{aligned} \tag{19}$$

Thus, from (18), (19) we can conclude that the limit does not exist.

The example (15) with $\limsup x^{-1}\log\tilde{F}^*(x) \neq \liminf x^{-1}\log\tilde{F}^*(x)$ is in a sense pathological, hence there is a possibility that we can guarantee the existence of the limit by posing appropriate weak conditions.

In fact, we have the following theorem, that is the main theorem of this article.

THEOREM 1 *For a non-negative random variable X, let $F(x) \equiv P(X \leq x)$ be the probability distribution function of X and $\tilde{F}(x) \equiv P(X > x)$ be the tail probability of X. Denote by σ_0 the abscissa of convergence of the Laplace–Stieltjes transform $\phi(s)$ of $F(x)$. If $-\infty < \sigma_0 < 0$ and the singularities of $\phi(s)$ on the axis of convergence $\Re s = \sigma_0$ are only a finite number of poles, then we have*

$$\lim_{x\to\infty}\frac{1}{x}\log\tilde{F}(x) = \sigma_0. \tag{20}$$

Remark 1 The assumption of Theorem 1 implies that there exists an open neighborhood U of $\Re s = \sigma_0$ and $\phi(s)$ is analytic on U except for the finite number of poles on $\Re s = \sigma_0$.

Remark 2 The real point $s = \sigma_0$ of the axis of convergence is always a singularity of $\phi(s)$ by Widder [5], p. 58, Theorem 5b (see Appendix). Hence, under the assumption in Theorem 1, $s = \sigma_0$ is a pole of $\phi(s)$.

In the case of discrete type random variable X, the following theorem is shown in [4], which corresponds to Theorem 1.

THEOREM (Nakagawa [4]) *Let X be a discrete type random variable taking non-negative integral values. The probability function of X is denoted by $\pi_n = P(X = n)$, $n = 0, 1, \ldots,$ and the tail probability by $\Pi_n \equiv P(X > n) = \sum_{j>n}\pi_j$. Consider the probability generating function $\pi(z)$ of $\{\pi_n\}$. If the radius of convergence r of $\pi(z)$ satisfies $1 < r < \infty$ and the singularities of $\pi(z)$ on the circle of convergence are only a finite number of poles, then we have*

$$\lim_{n\to\infty}\frac{1}{n}\log\Pi_n = -\log r.$$

The proof of this theorem depends mainly on the following fact for power series: if a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges in $|z| < R_1$ and is continued analytically to a function analytic in $|z| < R_2$ with $R_2 > R_1$, then the power series $\sum_{n=0}^{\infty} a_n z^n$ converges in $|z| < R_2$. However, for Laplace transforms, analogous to this property does not hold. In fact, even if the integral

$$\varphi(s) = \int_0^\infty e^{-sx}\mu(x)dx \tag{21}$$

converges in $\Re s > \sigma_1$ and $\varphi(s)$ is continued analytically to a function analytic in $\Re s > \sigma_2$ with $\sigma_2 < \sigma_1$, the integral (21) does not necessarily converge in $\Re s > \sigma_2$. For instance, the abscissa of convergence of the integral

$$\varphi(s) = \int_0^\infty e^{-sx}e^x\sin(e^x)dx$$

is $\sigma = 0$, nevertheless $\varphi(s)$ is continued analytically to a function analytic in the whole finite plane $|s| < \infty$ ([5], p.58). This is a large difference between power series and Laplace transform. In general, in Laplace transform, we cannot discuss the convergence of the integral in the left half-plane of the axis of convergence. However, we may be able to discuss the convergence on $\Re s = \sigma_0$ with the knowledge of convergence in $\Re s > \sigma_0$. For this purpose, we will apply Tauberian theorems.

4. Application of Tauberian theorems

4.1. *Abelian theorems*

First, we introduce a general form of Abelian theorem for Laplace transform.

A function $\mu(x)$ of bounded variation in $[a, b]$ is said to be *normalized* if it satisfies

$$\mu(a) = 0,$$
$$\mu(x) = \frac{1}{2}\big(\mu(x+) + \mu(x-)\big), \quad a < x < b.$$

Abelian Theorem (Widder [5], p. 183, Cor. 1c.) *For a normalized function $\mu(x)$ of bounded variation in $[0, R]$ for every $R > 0$, if the integral*

$$\varphi(s) = \int_0^\infty e^{-sx} d\mu(x)$$

exists for $s > 0$, then we have

$$\liminf_{x\to\infty} \mu(x) \le \liminf_{s\to 0+} \varphi(s) \le \limsup_{s\to 0+} \varphi(s) \le \limsup_{x\to\infty} \mu(x), \tag{22}$$

$$\liminf_{x\to 0+} \mu(x) \le \liminf_{s\to\infty} \varphi(s) \le \limsup_{s\to\infty} \varphi(s) \le \limsup_{x\to 0+} \mu(x). \tag{23}$$

4.2. *Tauberian theorems*

Contrary to Abelian theorems, Tauberian theorems give sufficient conditions for the existence of $\lim_{x\to\infty} \mu(x)$. The following is one of the well-known Tauberian theorems.

Tauberian Theorem (Widder [5], p. 187, Theorem 3b) *For a normalized function $\mu(x)$ of bounded variation in $[0, R]$ for every $R > 0$, let the integral*

$$\varphi(s) = \int_0^\infty e^{-sx} d\mu(x)$$

exist for $s > 0$ and let $\lim_{s\to 0+} \varphi(s) = A$. Then

$$\lim_{x\to\infty} \mu(x) = A$$

holds if and only if

$$\int_0^x t d\mu(t) = o(x), \quad x \to \infty$$

If we consider only a special type of function $\mu(x)$, then a weaker sufficient condition may be enough to describe the asymptotic behavior of $\mu(x)$. The following Tauberian theorem by Ikehara is for a non-negative and non-decreasing function. This Ikehara's Tauberian theorem could be used for the proof of the prime-number theorem [5].

Theorem (Ikehara [1], see also Widder [5], p. 233, Theorem 17) *Let $\mu(x)$ be a non-negative and non-decreasing function defined in $0 \le x < \infty$, and let the integral*

$$\psi(s) = \int_0^\infty e^{-sx}\mu(x)dx, \; s = \sigma + i\tau$$

converge for $\Re s > 1$. If for some constant α and some function $g(\tau)$

$$\lim_{\sigma\to 1+}\left(\psi(s) - \frac{\alpha}{s-1}\right) = g(\tau) \tag{24}$$

uniformly in every finite interval of τ, then we have

$$\lim_{x\to\infty} e^{-x}\mu(x) = \alpha.$$

Remark 3 Suppose $s = 1$ is a pole of $\psi(s)$ of order 1 and α the residue of $\psi(s)$ at $s = 1$. Further, if $\psi(s)$ has no other singularities than $s = 1$ on the axis of convergence $\Re s = 1$, then the condition (24) holds. More generally, suppose that the singularities of $\psi(s)$ on the axis of convergence $\Re s = 1$ are only a finite number of poles $s_j, j = 1, \ldots, m$. Denoting by $\psi_j(s)$ the principal part of $\psi(s)$ at the pole s_j, then similarly to (24),

$$\lim_{\sigma\to 1+}\left(\psi(s) - \sum_{j=1}^m \psi_j(s)\right) = g(\tau) \tag{25}$$

uniformly in every finite interval of τ.

5. Preliminary for proof of theorem 1

In this section, we will prepare some lemmas for the proof of Theorem 1.

First of all, we define

$$\delta(x) \equiv \frac{1}{\sqrt{2\pi}}\left(\frac{\sin x/2}{x/2}\right)^2, \quad -\infty < x < \infty,$$

and for any $\lambda > 0$,

$$\begin{aligned} k_\lambda(x) &\equiv 2\lambda\delta(2\lambda x) \\ &= \frac{2\lambda}{\sqrt{2\pi}}\left(\frac{\sin \lambda x}{\lambda x}\right)^2, \quad -\infty < x < \infty. \end{aligned}$$

For integers $D \geq 1$ and $N \geq (D+1)/2$, we see

$$\int_0^\infty k_\lambda^N(x-t)t^{D-1}dt < \infty, \qquad -\infty < \forall x < \infty,$$

where $k_\lambda^N(x-t)$ is the Nth power of $k_\lambda(x-t)$.

Next, for any $\lambda > 0$, define

$$K_\lambda(x) \equiv \begin{cases} 1 - \left|\dfrac{x}{2\lambda}\right|, & |x| \leq 2\lambda, \\ 0, & |x| > 2\lambda. \end{cases}$$

It follows that K_λ is the Fourier transform of k_λ, i.e., $K_\lambda = \mathcal{F}(k_\lambda)$ or

$$K_\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} k_\lambda(t)\, dt. \tag{26}$$

Writing $K_\lambda^{*N} = K_\lambda * \cdots * K_\lambda$ (N-fold convolution), we have $K_\lambda^{*N} = \mathcal{F}(k_\lambda^N)$. The support of K_λ^{*N} is denoted by $[-\Lambda,\ \Lambda]$, where $\Lambda = 2N\lambda$.

5.1. *On a slowly increasing function*

A function $f(x)$ defined in $0 \leq x < \infty$ is *slowly increasing* if it satisfies

$$\limsup_{x\to\infty, \eta\to 0+} \left\{ f(x+\eta) - f(x) \right\} \leq 0.$$

LEMMA 5 *Let $f(x)$ be a slowly increasing function. Given $\epsilon > 0$ and an increasing sequence $\{u_n\}_{n=0}^\infty$ with $u_n \to \infty$, we assume $f(u_n) > \epsilon$ (resp. $f(u_n) < -\epsilon$), $n = 1, 2, \ldots$ Then, there exist $\eta > 0$ and infinitely many non-overlapping intervals $\{J_n = (x_n - \eta, x_n + \eta)\}$ with*

$$f(x) > \frac{\epsilon}{2} \ \left(\text{resp. } f(x) < -\frac{\epsilon}{2}\right), \quad x \in J_n, \quad n = 1, 2, \ldots.$$

Proof: First, consider the case $f(u_n) > \epsilon$. For u_n, let us define $v_n = \sup\{v \mid v < u_n, f(v) < \epsilon/2\}$. Suppose the claim were not true, then for any $\eta > 0$ and sufficiently large all n,

$$0 \leq u_n - v_n < 2\eta. \tag{27}$$

By the definition of v_n, we have

$$f(u_n) - f(v_n) > \frac{\epsilon}{2}. \tag{28}$$

From (27), (28), we have

$$\begin{aligned} \limsup_{x\to\infty, \eta\to 0+} \{f(x+\eta) - f(x)\} &\geq \limsup_{n\to\infty} \{f(u_n) - f(v_n)\} \\ &\geq \frac{\epsilon}{2}, \end{aligned}$$

which contradicts $f(x)$ being slowly increasing. The case $f(u_n) < -\epsilon$ can be proved in a similar way. ■

We have the following lemma.

Lemma 6 *Let $f(x)$ be a bounded and slowly increasing function. Let D and N be integers with $D \geq 1$ and $N \geq (D+1)/2$. If*

$$\lim_{x\to\infty} \int_0^\infty k_\lambda^N(x-t)t^{D-1}f(t)\,dt = 0 \tag{29}$$

holds for any $\lambda > 0$, then we have

$$\lim_{x\to\infty} f(x) = 0. \tag{30}$$

Proof Suppose (30) were not true. Then there should exist $\epsilon > 0$ and an increasing sequence $\{u_n\}$ with $u_n \to \infty$ such that $f(u_n) > \epsilon$ or $f(u_n) < -\epsilon$. Assume $f(u_n) > \epsilon$. The other case can be handled in the same way.

From $f(u_n) > \epsilon$ and Lemma 5, there exists a sequence of intervals $\{J_n = (x_n - \eta, x_n + \eta)\}$ with

$$f(x) > \frac{\epsilon}{2}, \quad x \in J_n, \quad n = 1, 2, \ldots.$$

Hence, we have

$$\begin{aligned}
\int_0^\infty k_\lambda^N(x_n-t)t^{D-1}f(t)dt &= \int_0^{x_n-\eta} + \int_{x_n-\eta}^{x_n+\eta} + \int_{x_n+\eta}^\infty k_\lambda^N(x_n-t)t^{D-1}f(t)dt \\
&\geq \frac{\epsilon}{2}\int_{x_n-\eta}^{x_n+\eta} k_\lambda^N(x_n-t)t^{D-1}dt - \sup_{0\leq x<\infty}|f(x)| \\
&\quad \times \left\{\int_0^{x_n-\eta} + \int_{x_n+\eta}^\infty k_\lambda^N(x_n-t)t^{D-1}dt\right\}.
\end{aligned} \tag{31}$$

Each term in (31) can be evaluated as follows:

$$\begin{aligned}
\int_{x_n-\eta}^{x_n+\eta} k_\lambda^N(x_n-t)t^{D-1}dt &= \int_{-\eta}^{\eta} k_\lambda^N(t)(x_n-t)^{D-1}dt \\
&= (2\lambda)^{N-1}\int_{-2\lambda\eta}^{2\lambda\eta} \delta^N(t)\Big(x_n - \frac{t}{2\lambda}\Big)^{D-1}dt \\
&= (2\lambda)^{N-1}x_n^{D-1}\left\{\int_{-2\lambda\eta}^{2\lambda\eta}\delta^N(t)dt + \theta_1(x_n,\lambda)\right\},
\end{aligned} \tag{32}$$

where $\theta_1(x_n, \lambda) \to 0$ as $x_n, \lambda \to \infty$, by the binomial expansion of $\big(x_n - t/2\lambda\big)^{D-1}$.

Similarly, we have

$$\int_0^{x_n-\eta} k_\lambda^N(x_n-t)t^{D-1}dt = (2\lambda)^{N-1}x_n^{D-1}\theta_2(x_n,\lambda), \tag{33}$$

and

$$\int_{x_n+\eta}^{\infty} k_\lambda^N(x_n-t)t^{D-1}dt = (2\lambda)^{N-1}x_n^{D-1}\theta_3(x_n,\lambda), \tag{34}$$

where $\theta_2(x_n,\lambda)\to 0, \theta_3(x_n,\lambda)\to 0$ as $x_n,\lambda\to\infty$.

Write $C\equiv\int_0^\infty \delta^N(t)dt<\infty$ and $M\equiv\sup_{0\le x<\infty}|f(x)|<\infty$, then there exists $\tilde{C}>0$ such that for sufficiently small $\xi_1,\xi_2>0$ and large λ, from (31)–(34), we have

$$\begin{aligned}\int_0^\infty k_\lambda^N(x_n-t)t^{D-1}f(t)dt &\ge \left\{\frac{\epsilon}{2}(C-\xi_1)-M\xi_2\right\}(2\lambda)^{N-1}x_n^{D-1}\\ &\ge \tilde{C},\end{aligned}$$

but this contradicts the assumption of the lemma (29).

5.2. *Laplace transform with finite number of poles on its axis of convergence*

Let $\mu(x)$ be a non-negative and non-increasing function in $x\ge 0$, and $\mu(x)\in L^1[0,R]$ for any $R>0$. Consider the Laplace transform

$$\psi(s)=\int_0^\infty e^{-sx}\mu(x)\,dx.$$

Let σ_0 be the abscissa of convergence of $\psi(s)$ with $-\infty<\sigma_0<0$. Assume the singularities of $\psi(s)$ on the axis of convergence $\Re s=\sigma_0$ are only a finite number of poles $s_j=\sigma_0+i\tau_j$, $j=1,\ldots,m$. Write $\psi_j(s)$ as the principal part of $\psi(s)$ at the pole s_j, and $A_j(x)$ as the inverse Laplace transform of $\psi_j(s)$, $j=1,\ldots,m$. The order of the pole s_j is denoted by d_j and let $D=\max_{1\le j\le m} d_j$.

Define functions $b(x)$ and $B(x)$ for $x\ge 0$ as follows:

$$b(x)=e^{-\sigma_0 x}x^{-D+1}\mu(x),\ x\ge 0, \tag{35}$$

$$B(x)=e^{-\sigma_0 x}x^{-D+1}\sum_{j=1}^m A_j(x),\ x\ge 0. \tag{36}$$

We will investigate the properties of $b(x)$ and $B(x)$.

We have the following lemma.

LEMMA 7 *Let N be an integer with $N \geq (D+1)/2$. If*

$$\frac{1}{x^{D-1}}\int_0^\infty k_\lambda^N(x-t)t^{D-1}\{b(t)-B(t)\}dt$$

is bounded in $x \geq 0$ for some $\lambda > 0$, then $b(x) - B(x)$ is bounded.

Proof Since the principal part $\psi_j(s)$ at the pole s_j is represented as

$$\psi_j(s) = \sum_{l=1}^{d_j} \frac{\alpha_{jl}}{(s-s_j)^l}, \ \alpha_{jl} \in \mathrm{C}, \ \alpha_{jd_j} \neq 0,$$

we have

$$A_j(x) = \sum_{l=1}^{d_j} \frac{\alpha_{jl}}{(l-1)!} x^{l-1} e^{s_j x}.$$

Thus, we have from (36)

$$B(x) = \sum_{j=1}^{m}\sum_{l=1}^{d_j} \frac{\alpha_{jl}}{(l-1)!} x^{l-D} e^{i\tau_j x}. \tag{37}$$

Since $l - D \leq 0$, $B(x)$ is bounded in $x \geq 0$.

Next, we will show that $b(x)$ is bounded. We have

$$\begin{aligned}\int_0^\infty k_\lambda^N(x-t)t^{D-1}B(t)dt &= \int_{-x}^\infty k_\lambda^N(t)(x+t)^{D-1}B(x+t)\,dt \\ &= \sum_{n=0}^{D-1}\binom{D-1}{n} x^{D-1-n}\int_{-x}^\infty k_\lambda^N(t)t^n B(x+t)\,dt. \end{aligned}\tag{38}$$

Since $B(x)$ is bounded and

$$\left|\int_{-x}^\infty k_\lambda^N(t)t^{D-1}dt\right| < \infty,$$

we see from (38) that there exist $E_1 > 0$ and $x_1 > 0$ with

$$\frac{1}{x^{D-1}}\left|\int_0^\infty k_\lambda^N(x-t)t^{D-1}B(t)dt\right| < E_1, \quad x > x_1. \tag{39}$$

Noting $b(x) \geq 0$, we have from the assumption of the lemma and (39),

$$\begin{aligned}
&\frac{1}{x^{D-1}}\int_0^{\infty} k_\lambda^N(x-t)t^{D-1}b(t)\,dt \\
&\quad \leq \frac{1}{x^{D-1}}\left|\int_0^{\infty} k_\lambda^N(x-t)t^{D-1}\{b(t)-B(t)\}dt\right| + \frac{1}{x^{D-1}}\left|\int_0^{\infty} k_\lambda^N(x-t)t^{D-1}B(t)dt\right| \\
&\quad < E_2, \ x > x_2. \qquad (40)
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
\frac{1}{x^{D-1}}\int_0^{\infty} k_\lambda^N(x-t)t^{D-1}b(t)\,dt &= \frac{1}{x^{D-1}}\int_{-x}^{\infty} k_\lambda^N(t)(x+t)^{D-1}b(x+t)\,dt \\
&= \frac{(2\lambda)^{N-1}}{x^{D-1}}\int_{-2\lambda x}^{\infty} \delta^N(t)\Big(x+\frac{t}{2\lambda}\Big)^{D-1} b\Big(x+\frac{t}{2\lambda}\Big)\,dt,
\end{aligned}$$

hence it follows from (40) that

$$\frac{(2\lambda)^{N-1}}{x^{D-1}}\int_{-2\lambda x}^{\infty} \delta^N(t)\Big(x+\frac{t}{2\lambda}\Big)^{D-1} b\Big(x+\frac{t}{2\lambda}\Big)\,dt \ < E_2, \ x > x_2. \qquad (41)$$

The integrand in (41) is non-negative in $[-2\lambda x, \infty)$ and $[-\sqrt{\lambda}, \sqrt{\lambda}] \subset [-2\lambda x, \infty)$, hence we have

$$\frac{(2\lambda)^{N-1}}{x^{D-1}}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \delta^N(t)\Big(x+\frac{t}{2\lambda}\Big)^{D-1} b\Big(x+\frac{t}{2\lambda}\Big)dt \ < E_2, \ x > x_2,$$

and thus

$$\frac{(2\lambda)^{N-1}}{x^{D-1}}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \delta^N(t)\Big(x-\frac{1}{2\sqrt{\lambda}}+\frac{t}{2\lambda}\Big)^{D-1} b\Big(x-\frac{1}{2\sqrt{\lambda}}+\frac{t}{2\lambda}\Big)dt \ < E_2, \ x > x_3. \qquad (42)$$

Since $\mu(x)$ is non-increasing, we have for $-\sqrt{\lambda} \leq t \leq \sqrt{\lambda}$,

$$\begin{aligned}
\Big(x-\frac{1}{2\sqrt{\lambda}}+\frac{t}{2\lambda}\Big)^{D-1} b\Big(x-\frac{1}{2\sqrt{\lambda}}+\frac{t}{2\lambda}\Big) &= e^{-\sigma_0\left(x-(1/\sqrt{2\lambda})+(t/2\lambda)\right)}\mu\Big(x-\frac{1}{2\sqrt{\lambda}}+\frac{t}{2\lambda}\Big) \\
&\geq e^{-\sigma_0\left(x-(1/\sqrt{2\lambda})+(t/2\lambda)\right)}\mu(x),
\end{aligned}$$

and from (42)

$$e^{-\sigma_0 x}x^{-D+1}\mu(x)\frac{(2\lambda)^{N-1}}{\sqrt{2\pi}}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} e^{\sigma_0((1/2\sqrt{\lambda})-(t/2\lambda))}\delta^N(t)\,dt \ < E_2, \ x > x_3.$$

Therefore, we finally have

$$b(x) < E_2\Big(\frac{(2\lambda)^{N-1}}{\sqrt{2\pi}}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} e^{\sigma_0((1/2\sqrt{\lambda})-(t/2\lambda))}\delta^N(t)\,dt\Big)^{-1},\ x > x_3,$$

which shows that $b(x)$ is bounded. ■

LEMMA 8 *Under the assumption of Lemma 7, $b(x) - B(x)$ is slowly increasing.*

Proof It is readily seen that the sum of two slowly increasing functions is slowly increasing. We show that both $b(x)$ and $-B(x)$ are slowly increasing. Since $B(x)$ has bounded derivative, $-B(x)$ is slowly increasing.

Next, we prove that $b(x)$ is slowly increasing. For $x > 0$ and $\eta > 0$, we have

$$\begin{aligned} b(x+\eta) - b(x) &= e^{-\sigma_0(x+\eta)}(x+\eta)^{-D+1}\mu(x+\eta) - e^{-\sigma_0 x}x^{-D+1}\mu(x) \\ &\le b(x)\left\{e^{-\sigma_0\eta}\frac{x^{D-1}}{(x+\eta)^{D-1}} - 1\right\} \end{aligned}$$

because $\mu(x)$ is non-increasing. Since $b(x)$ is bounded by Lemma 7, we have

$$\begin{aligned} \limsup_{x\to\infty,\,\eta\to 0+}\{b(x+\eta) - b(x)\} &\le \limsup_{x\to\infty,\,\eta\to 0+} b(x)\left\{e^{-\sigma_0\eta}\frac{x^{D-1}}{(x+\eta)^{D-1}} - 1\right\} \\ &= 0, \end{aligned}$$

which shows that $b(x)$ is slowly increasing.

5.3. *Extension of Ikehara's theorem*

The following theorem is an extension of Ikehara's theorem to the case that $\mu(x)$ is a non-negative and non-increasing function and its Laplace transform has a finite number of poles on the axis of convergence.

THEOREM 2 *Let $\mu(x)$ be a non-negative and non-increasing function in $x \ge 0$, and $\mu(x) \in L^1[0, R]$ for any $R > 0$. Let σ_0 be the abscissa of convergence of the Laplace transform*

$$\psi(s) = \int_0^\infty e^{-sx}\mu(x)dx \tag{43}$$

with $-\infty < \sigma_0 < 0$. Assume the singularities of $\psi(s)$ on the axis of convergence $\Re s = \sigma_0$ are only a finite number of poles $s_j = \sigma_0 + i\tau_j$, $j = 1, \ldots, m$. Write $\psi_j(s)$ as the principal part of $\psi(s)$ at s_j and $A_j(x)$ as the inverse Laplace transform of $\psi_j(s)$, $j = 1, \ldots, m$. The order of the pole s_j is denoted by d_j and let $D = \max_{1\le j\le m} d_j$. Then we have

$$\lim_{x\to\infty}\left\{e^{-\sigma_0 x}x^{-D+1}\Big(\mu(x) - \sum_{j=1}^{m} A_j(x)\Big)\right\} = 0. \tag{44}$$

Remark 4 From Korevaar [2], p. 495, Theorem 9.2 (see Appendix), if

$$e^{-\sigma_0 x}x^{-D+1}\big(\mu(x)-\sum_{j=1}^{m}A_j(x)\big)$$

is differentiable and its derivative is bounded from below, then we see that (44) holds.

Proof Consider the functions $b(x)$ and $B(x)$ defined in (35), (36). For any $\epsilon > 0$ and an integer N with $N \geq (D+1)/2$, define

$$I_\epsilon(x) \equiv \frac{1}{\sqrt{2\pi}}\int_0^\infty k_\lambda^N(x-t)t^{D-1}\{b(t)-B(t)\}e^{-\epsilon t}dt.$$

Since $K_\lambda^{*N} = \mathcal{F}(k_\lambda^N)$, by Fubini's theorem, we have

$$\begin{aligned}
I_\epsilon(x) &= \frac{1}{2\pi}\int_0^\infty t^{D-1}\{b(t)-B(t)\}e^{-\epsilon t}dt\int_{-\Lambda}^{\Lambda}K_\lambda^{*N}(\tau)e^{-i\tau(x-t)}d\tau\\
&= \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}K_\lambda^{*N}(\tau)e^{-ix\tau}d\tau\int_0^\infty t^{D-1}\{b(t)-B(t)\}e^{-(\epsilon-i\tau)t}dt\\
&= \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}K_\lambda^{*N}(\tau)e^{-ix\tau}d\tau\left\{\psi(\sigma_0+\epsilon-i\tau)-\sum_{j=1}^{m}\psi_j(\sigma_0+\epsilon-i\tau)\right\}. \qquad (45)
\end{aligned}$$

By the assumption of theorem and Remark 3, there exists a function $g(\tau)$ and

$$\lim_{\epsilon\to 0+}\left\{\psi(\sigma_0+\epsilon-i\tau)-\sum_{j=1}^{m}\psi_j(\sigma_0+\epsilon-i\tau)\right\} = g(\tau) \qquad (46)$$

uniformly in the finite interval $[-\Lambda,\ \Lambda]$, which is the support of K_λ^{*N} (see the definition after (26)). Therefore, we can change the order of the limit and the integration in (45) to obtain

$$\lim_{\epsilon\to 0+} I_\epsilon(x) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}K_\lambda^{*N}(\tau)e^{-ix\tau}g(\tau)\,d\tau. \qquad (47)$$

Next, consider

$$I_{\epsilon,B}(x) \equiv \frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}B(t)e^{-\epsilon t}dt.$$

From (37), we have

$$I_{\epsilon,B}(x) = \frac{1}{2\pi}\sum_{j=1}^{m}\sum_{l=1}^{d_j}\frac{\alpha_{jl}}{(l-1)!}\int_0^\infty k_\lambda^N(x-t)t^{l-1}e^{i\tau_j t}e^{-\epsilon t}dt. \qquad (48)$$

Since $N \geq (D+1)/2$, the infinite integral

$$\int_0^\infty k_\lambda^N(x-t)t^{l-1}e^{i\tau_j t}\,dt$$

converges, thus we have by (22) in the Abelian theorem

$$\lim_{\epsilon\to 0+}\int_0^\infty k_\lambda^N(x-t)t^{l-1}e^{i\tau_j t}e^{-\epsilon t}dt = \int_0^\infty k_\lambda^N(x-t)t^{l-1}e^{i\tau_j t}\,dt.$$

Hence we have by (48)

$$\lim_{\epsilon\to 0+} I_{\epsilon,B}(x) = \frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}B(t)\,dt. \tag{49}$$

Moreover, consider

$$I_{\epsilon,b}(x) \equiv \frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}b(t)e^{-\epsilon t}\,dt.$$

By (47), (49), we have

$$\lim_{\epsilon\to 0+} I_{\epsilon,b}(x) = \lim_{\epsilon\to 0+}\left\{I_\epsilon(x) + I_{\epsilon,B}(x)\right\} < \infty. \tag{50}$$

Since $b(t) \geq 0$, we have

$$\frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}b(t)dt \leq \infty,$$

but if the integral is ∞, then from (22),

$$\begin{aligned}\lim_{\epsilon\to 0+} I_{\epsilon,b(x)} &\geq \liminf_{R\to\infty}\frac{1}{2\pi}\int_0^R k_\lambda^N(x-t)t^{D-1}b(t)dt\\ &= \infty\end{aligned}$$

which contradicts (50). Hence we have

$$\frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}b(t)\,dt < \infty.$$

By (22), we have

$$\begin{aligned}\lim_{\epsilon\to 0+} I_\epsilon(x) &= \lim_{\epsilon\to 0+} I_{\epsilon,b}(x) - \lim_{\epsilon\to 0+} I_{\epsilon,B}(x)\\ &= \frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}\left\{b(t)-B(t)\right\}dt.\end{aligned} \tag{51}$$

Thus by (47), (51), it follows that

$$\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} K_\lambda^{*N}(\tau)e^{-ix\tau}g(\tau)d\tau = \frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}\{b(t)-B(t)\}dt,$$

and hence by the Riemann–Lebesgue theorem

$$\lim_{x\to\infty}\frac{1}{2\pi}\int_0^\infty k_\lambda^N(x-t)t^{D-1}\{b(t)-B(t)\}dt = 0 \tag{52}$$

holds for any $\lambda > 0$. So, the assumption of Lemma 7 is satisfied. Since $b(x) - B(x)$ is bounded and slowly increasing by Lemmas 7, 8, we have by (52) and Lemma 6, $\lim_{x\to\infty}\{b(x) - B(x)\} = 0$ or

$$\lim_{x\to\infty}\left\{e^{-\sigma_0 x}x^{-D+1}\left(\mu(x) - \sum_{j=1}^m A_j(x)\right)\right\} = 0$$

■

5.4. *Estimation of $B(x)$ from below*

Our goal is to examine the exponential decay of the tail probability $\tilde{F}(x)$. Replace $\mu(x)$ in Theorem 2 by $\tilde{F}(x)$. The basic approach to this problem is to approximate $\tilde{F}(x)$ by the inverse Laplace transform $\sum_{j=1}^m A_j(x)$ of the rational function $\sum_{j=1}^m \psi_j(s)$. To this end, we first estimate $B(x)$ from below.

LEMMA 9 *Let τ_j, $j = 1, \ldots, m$, be distinct real numbers and d_j, $j = 1, \ldots, m$, be integers with $d_j \geq 1$. Let α_{jl}, $l = 1, \ldots, d_j$, $j = 1, \ldots, m$, be complex numbers with $\alpha_{jd_j} \neq 0$. Write $D = \max_{1\leq j\leq m} d_j$. For $x \geq 0$, define*

$$B(x) = \sum_{j=1}^m \sum_{l=1}^{d_j} \tilde{\alpha}_{jl}x^{l-D}e^{i\tau_j x}, \quad \tilde{\alpha}_{jl} \equiv \frac{\alpha_{jl}}{(l-1)!}, \ x \geq 0. \tag{53}$$

Then there exists $T > 0$, $\epsilon = \epsilon(T) > 0$, and $x_0 = x_0(T)$, such that for any $x > x_0$ at least one of $|B(x)|, |B(x+T)|, \ldots, |B(x+(m-1)T)|$ is greater than ϵ.

Proof Since $\tau_j \neq \tau_{j'}$, $j \neq j'$, there exists $T > 0$ with $\tau_j T \not\equiv \tau_{j'} T \bmod 2\pi$, $j \neq j'$. For this T, we will show that there exists $\epsilon > 0$ such that at least one of $|B(x)|, \ldots, |B(x+(m-1)T)|$ is greater than ϵ for any $x > x_0$. If not, then for any $\epsilon > 0$ there exists a sufficiently large x with

$$|B(x+nT)| \leq \epsilon, \ n = 0, 1, \ldots, m-1. \tag{54}$$

Now, define

$$\beta_j(x) = \sum_{l=1}^{d_j} \tilde{\alpha}_{jl}x^{l-D}, \quad j = 1, \ldots, m. \tag{55}$$

Denote by m^* the number of indices j that attains $D = \max_{1 \le j \le m} d_j$. Without loss of generality, we can assume $d_1 = \cdots = d_{m^*} = D$. Then by (55) we can write

$$\beta_j(x) = \begin{cases} \tilde{\alpha}_{jD} + \kappa_j(x), & j = 1, \ldots, m^* \\ \kappa_j(x), & j = m^* + 1, \ldots, m, \end{cases}$$

where $\kappa_j(x)$ is a function with $\lim_{x \to \infty} \kappa_j(x) = 0$, $j = 1, \ldots, m$. Defining $\kappa(x) = \sum_{j=1}^{m} \kappa_j(x)$, we have from (53), (55)

$$\begin{aligned} B(x) &= \sum_{j=1}^{m} \beta_j(x) e^{i\tau_j x} \\ &= \sum_{j=1}^{m^*} \tilde{\alpha}_{jD} e^{i\tau_j x} + \kappa(x). \end{aligned}$$

Therefore we have

$$B(x + nT) = \sum_{j=1}^{m^*} \tilde{\alpha}_{jD} e^{i\tau_j (x+nT)} + \kappa(x + nT), \quad n = 0, 1, \ldots, m-1. \tag{56}$$

The matrix representation of (56) is

$$\begin{pmatrix} B(x) \\ B(x+T) \\ \vdots \\ B(x+(m-1)T) \end{pmatrix} = \begin{pmatrix} e^{i\tau_1 x} & e^{i\tau_2 x} & \ldots & e^{i\tau_m x} \\ e^{i\tau_1 (x+T)} & e^{i\tau_2 (x+T)} & \ldots & e^{i\tau_m (x+T)} \\ & \ldots & & \\ e^{i\tau_1 (x+(m-1)T)} & e^{i\tau_2 (x+(m-1)T)} & \ldots & e^{i\tau_m (x+(m-1)T)} \end{pmatrix}$$
$$\times \begin{pmatrix} \tilde{\alpha}_{1D} \\ \vdots \\ \tilde{\alpha}_{m^* D} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \kappa(x) \\ \kappa(x+T) \\ \vdots \\ \kappa(x+(m-1)T) \end{pmatrix}. \tag{57}$$

Writing V as the matrix in the right-hand side of (57), we have

$$\begin{aligned} \det V &= \prod_{j=1}^{m} e^{i\tau_j x} \times \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ e^{i\tau_1 T} & e^{i\tau_2 T} & \ldots & e^{i\tau_m T} \\ & \ldots & & \\ e^{i\tau_1^{(m-1)T}} & e^{i\tau_2 (m-1)T} & \ldots & e^{i\tau_m (m-1)T} \end{pmatrix} \\ &= \prod_{j=1}^{m} e^{i\tau_j x} \times (-1)^{m(m-1)/2} \prod_{j<j'} \left(e^{i\tau_j T} - e^{i\tau_{j'} T}\right). \text{ (Vandermonde determinant)} \end{aligned}$$

Since $e^{i\tau_j T} \neq e^{i\tau_{j'} T}$ by $\tau_j T \not\equiv \tau_{j'} T \bmod 2\pi,\ j \neq j'$, we have

$$|\det V| = \prod_{j<j'} |e^{i\tau_j T} - e^{i\tau_{j'} T}| > 0.$$

Denoting by Δ_{jn} the cofactor of V, we have by (57)

$$\tilde{\alpha}_{jD} = \frac{1}{\det V} \sum_{n=0}^{m-1} \Delta_{jn}\big(B(x+nT) - \kappa(x+nT)\big),\ j = 1, \ldots, m^*. \tag{58}$$

For any $\epsilon_1 > 0$ there exists a sufficiently large x_1 such that $|\kappa(x)| < \epsilon_1$ holds for any $x > x_1$. Since Δ_{jn} is $(-1)^{j+n}$ multiplied by the sum of $(m-1)!$ complex numbers of absolute value 1, we have $|\Delta_{jn}| \leq (m-1)!$. Hence, by (54), (58)

$$|\tilde{\alpha}_{jD}| \leq \frac{m!}{|\det V|}(\epsilon + \epsilon_1),\ j = 1, \ldots, m^*.$$

Because $\epsilon,\ \epsilon_1$ are arbitrary, we have $\tilde{\alpha}_{jD} = 0$, but this is a contradiction. ∎

The following lemma is an immediate consequence of Lemma 9.

Lemma 10 *There exist* $L > 0,\ c = c(L) > 0,$ *and an increasing sequence* $\{x_\nu\}_{\nu=1}^{\infty}$ *with* $x_\nu \to \infty$ *which is contained in* $[x_0, \infty)$ *for* x_0 *sufficiently large, such that*

$$x_\nu - x_{\nu-1} \leq L, \tag{59}$$

$$|B(x_\nu)| \geq c, \nu = 1, 2, \ldots. \tag{60}$$

Proof Let $L = 2mT$ for m and T in the proof of Lemma 9. ∎

6. Proof of theorem 1

Based on the lemmas in the preceding sections, we will prove our main theorem.

Proof of Theorem 1 By Lemma 4,

$$\limsup_{x\to\infty} \frac{1}{x} \log \tilde{F}(x) = \sigma_0,$$

hence, we only need to prove

$$\liminf_{x\to\infty} \frac{1}{x} \log \tilde{F}(x) \geq \sigma_0.$$

Consider the increasing sequence $\{x_\nu\}$ with L and c which are given in Lemma 10.

Now, by Theorem 2, for any ϵ with $0 < \epsilon < c$, taking sufficiently large x_0, we have

$$\left| e^{-\sigma_0 x} x^{-D+1} \left\{ \tilde{F}(x) - \sum_{j=1}^{m} A_j(x) \right\} \right| < \epsilon, \; x > x_0.$$

Thus, it follows that

$$\begin{aligned} \tilde{F}(x) &> \Big| \sum_{j=1}^{m} A_j(x) \Big| - \epsilon e^{\sigma_0 x} x^{D-1} \\ &= e^{\sigma_0 x} x^{D-1} \big\{ |B(x)| - \epsilon \big\}, x > x_0. \end{aligned}$$

Particularly for $x = x_\nu$, by (60) we have

$$\tilde{F}(x_\nu) > e^{\sigma_0 x_\nu} x_\nu^{D-1} (c - \epsilon). \tag{61}$$

For sufficiently large any x, take the index ν with $x_{\nu-1} < x \leq x_\nu$. Then by (59), (61) and the decreasing property of $\tilde{F}(x)$, we have

$$\begin{aligned} \tilde{F}(x) &\geq \tilde{F}(x_\nu) \\ &> (c - \epsilon) e^{\sigma_0 x_\nu} x_\nu^{D-1} \\ &\geq (c - \epsilon) e^{\sigma_0 (x+L)} x^{D-1}. \end{aligned} \tag{62}$$

Therefore from (62), we have

$$\liminf_{x \to \infty} \frac{1}{x} \log \tilde{F}(x) \geq \sigma_0.$$

■

Theorem 1 gives a sufficient condition in which the number of poles on the axis of convergence $\Re s = \sigma_0$ is finite. But in a special case, there exists $\lim_{x \to \infty} x^{-1} \log \tilde{F}(x)$ even if the number of poles on $\Re s = \sigma_0$ is infinite. We show the following theorem.

THEOREM 3 *Let $\pi = \{\pi_n\}_{n=0}^{\infty}$ be a discrete type probability distribution and $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$ be the probability generating function of π. We assume that the radius of convergence r of $\pi(z)$ satisfies $1 < r < \infty$ and the singularities of $\pi(z)$ on the circle of convergence $|z| = r$ are only a finite number of poles. Let X be a continuous type random variable whose probability distribution function is defined by*

$$F(x) = \sum_{n \leq x} \pi_n, \; x \geq 0.$$

Then the tail probability $\tilde{F}(x)$ of X satisfies

$$\lim_{x \to \infty} \frac{1}{x} \log \tilde{F}(x) = -\log r.$$

Proof By Theorem 3 in Nakagawa [4]. ■

Remark 5 The Laplace–Stieltjes transform $\phi(s)$ of $F(x)$ in Theorem 3 satisfies

$$\begin{aligned}\phi(s) &= \int_0^\infty e^{-sx} dF(x)\\ &= \sum_{n=0}^\infty e^{-sn}\pi_n\\ &= \sum_{n=0}^\infty \pi_n z^n, z = e^{-s},\end{aligned}$$

so, by the change of variable $z = e^{-s}$, we see that $\phi(s)$ has infinitely many poles on the axis of convergence $\Re s = -\log r$.

Then, how about the case of X^* which was defined in (15)? For X^*, let $F^*(x)$ denote the probability distribution function of X^*, $\phi^*(s)$ the Laplace–Stieltjes transform of $F^*(x)$, and $\tilde{F}^*(x)$ the tail probability of X^*. Since the limit $\lim_{x\to\infty} x^{-1}\log\tilde{F}^*(x)$ does not exist, $\phi^*(s)$ may have different type of infinitely many singularities from those in Theorem 3. We will show that, by taking h as a special value, all the points on the axis of convergence $\Re s = \sigma_0$ are singularities of $\phi^*(s)$.

First, in the definition (9) of the sequence $\{c_n\}$, take h such that e^h is an integer. Then all the c_n are non-negative integers. For $\{c_n\}$, define $\gamma(x)$ by (10), and define $F^*(x)$ by

$$F^*(x) = 1 - e^{\sigma_0 x}\gamma(x), \quad \sigma_0 < 0, \quad x \geq 0.$$

Write $\tilde{F}^*(x) = 1 - F^*(x)$ and

$$\phi^*(s) = \int_0^\infty e^{-sx} dF^*(x), \ \Re s > \sigma_0.$$

Since $d\tilde{F}^*(x) = \sigma_0\tilde{F}^*(x)dx + e^{\sigma_0 x}d\gamma(x)$, we have by Lemma 2,

$$\begin{aligned}\phi^*(s) &= -\int_0^\infty e^{-sx} d\tilde{F}^*(x)\\ &= -\sigma_0\int_0^\infty e^{-sx}\tilde{F}^*(x)\,dx - \int_0^\infty e^{(\sigma_0-s)x}d\gamma(x)\\ &= \frac{\sigma_0}{s}\left\{\int_0^\infty e^{-sx}dF^*(x)dx - 1\right\} - \int_0^\infty e^{(\sigma_0-s)x}d\gamma(x)\\ &= \frac{\sigma_0}{s}(\phi^*(s)-1) - \int_0^\infty e^{(\sigma_0-s)x}d\gamma(x).\end{aligned} \tag{63}$$

Defining

$$\xi(s) = \int_0^\infty e^{(\sigma_0-s)x}d\gamma(x),$$

we have by (63)

$$\phi^*(s) = -\frac{\sigma_0 + s\xi(s)}{s - \sigma_0}. \tag{64}$$

Since we can write $\xi(s)$ as

$$\xi(s) = \sum_{n=0}^{\infty} e^{(\sigma_0 - s)c_n}(e^{-hc_{n-1}} - e^{-hc_n}),$$

and c_n is a non-negative integer, by the change of variable $e^{\sigma_0 - s} = z$, define $\Xi(z)$ as

$$\begin{aligned}\Xi(z) &\equiv \xi(s)\\ &= \sum_{n=0}^{\infty}(e^{-hc_{n-1}} - e^{-hc_n})z^{c_n}.\end{aligned}$$

Since the abscissa of convergence $\phi^*(s)$ is σ_0, that of $\xi(s)$ is also σ_0 by (64). Thus, the radius of convergence of $\Xi(z)$ is 1. The sequence $\{c_n\}$ is a gap sequence, i.e., $\liminf_{n\to\infty} c_{n+1}/c_n > 1$, hence all the points on the circle of convergence $|z| = 1$ are singularities of $\Xi(z)$ by Hadamard's gap theorem shown in Appendix. Therefore, by the correspondence $e^{\sigma_0 - s} = z$, all the points on the axis of convergence $\Re s = \sigma_0$ are singularities of $\xi(s)$, hence by (64) these are singularities of $\phi^*(s)$, too.

7. Conclusion

We investigated the exponential decay of the tail probability $P(X > x)$ of a continuous type random variable X based on the analytic properties of the Laplace–Stieltjes transform $\phi(s)$ of the probability distribution function $P(X \le x)$. We saw that the singularities of $\phi(s)$ on the axis of convergence play an essential role. For the proof of the exponential decay of $P(X > x)$, Ikehara's Tauberian theorem was extended and applied.

The exponential decay rate is determined by the abscissa of convergence σ_0 and $s = \sigma_0$ is always a singularity of $\phi(s)$. Through the proof of the main theorem, we believe the following conjecture is true.

Conjecture: If $-\infty < \sigma_0 < 0$ and $s = \sigma_0$ is a pole of $\phi(s)$, then the tail probability decays exponentially.

References

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[3] Markushevich, A.I., 1985, *Theory of Functions of a Complex Variable* (New York: Chelsea Publishing Company).

[4] Nakagawa, K., 2004, On the exponential decay rate of the tail of a discrete probability distribution. *Stochastic Models*, **20**(1), 31–42.

[5] Widder, D.V., 1941, *The Laplace Transform* (Princeton, NJ: Princeton University Press).

A. Citation of theorems

Theorem (Widder [5], p. 44, Theorem 2.4e) *If*

$$\int_0^{\infty} e^{-sx} df(x)$$

has a negative abscissa of convergence σ_0, *then* $f(\infty)$ *exists and*

$$\sigma_0 = \limsup_{x \to \infty} \frac{1}{x} \log |f(x) - f(\infty)|.$$

Theorem (Widder [5], p. 58, Theorem 5b) *If* $f(x)$ *is monotonic, then the real point of the axis of convergence of*

$$\phi(s) = \int_0^{\infty} e^{-sx} df(x)$$

is a singularity of $\phi(s)$.

Theorem (Widder [5], p. 40, Theorem 2.2b) *If*

$$\int_0^{\infty} e^{-sx} df(x)$$

converges for s *with* $\Re s = \sigma < 0$, *then* $f(\infty)$ *exists and*

$$f(x) - f(\infty) = o(e^{\sigma x}).$$

Theorem (Korevaar [2], p. 495, Theorem 9.2) *Let* $f(x)$ *be bounded from below for* $t \geq 0$ *and*

$$\varphi(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

converge for s *with* $\Re s > 0$. *If* $\varphi(s)$ *is continued analytically to the closed half-plane* $\Re s \geq 0$, *then* $\int_0^{\infty} f(x) dx$ *exists and is equal to* $\varphi(0)$.

Theorem (Hadamard's Gap Theorem, see [3]) *If a power series* $f(z) = \sum_{n=0}^{\infty} a_{c_n} z^{c_n}$ *satisfies*

$$\liminf_{n \to \infty} \frac{c_{n+1}}{c_n} > 1,$$

then all the points on the circle of convergence are singularities of $f(z)$.