PAPER Special Section on Information Theory and Its Applications

The Importance Sampling Simulation of MMPP/D/1 Queueing*

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SUMMARY We investigate an importance sampling (IS) simulation of MMPP/D/1 queueing to obtain an estimate for the survivor function $P(Q > q)$ of the queue length Q in the steady state. In [11], we studied the IS simulation of 2-state MMPP/D/1 queueing and obtained the optimal simulation distribution, but the mathematical fundation of the theory was not enough. In this paper, we construct a discrete time Markov chain model of the n-state MMPP/D/1 queueing and extend the results of [11] to the n-state MMPP/D/1. Based on the Markov chain model, we determine the optimal IS simulation distribution of the n-state MMPP/D/1 queueing by applying the large deviations theory, especially, the sample path large deviations theory. Then, we carry out IS simulation with the obtained optimal simulation distribution. Finally, we compare the simulation results of the IS simulation with the ordinary Monte Carlo (MC) simulation. We show that, in a typical case, the ratio of the computation time of the IS simulation to that of the MC simulation is about 10^{-7}, and the 95% confidence interval of the IS is slightly improved compared with the MC.

key words: *simulation, queueing, importance sampling, MMPP, large deviations theory*

1. Introduction

The importance sampling (IS) simulation technique has been exploited to have an accurate estimate for a very small probability that is not tractable by the ordinary Monte Carlo (MC) simulation.

In this paper, we investigate a simulation of the cell loss probability in ATM networks. According to a service category, the required cell loss probability standard at a multiplexer is less than 10^{-12}. In general, it is very difficult to obtain such a small probability by an ordinary MC simulation. Therefore, some kind of acceleration is necessary for the MC simulation.

The IS method is applied to this ATM queueing problem. We investigate an MMPP/D/1 queueing model to obtain an estimate for the survivor function $P(Q > q)$ of the queue length Q in the steady state. We first represent the MMPP/D/1 by a discrete time Markov chain model and determine the optimal simulation distribution by applying the large deviations theory, especially, the sample path large deviations theory, [1], [2]. Next, we carry out IS simulation with the obtained optimal simulation distribution using the notion of regenerative cycle and dynamic importance sampling technique [4], [11]. Finally, we compare simulation results of the IS simulation with the ordinary MC simulation to confirm the optimality of the obtained simulation distribution. We see that, in a typical case where the target probability is 10^{-12}, the ratio of the computation time of the IS simulation to that of the MC simulation is about 10^{-7}, and the 95% confidence interval of the IS estimate is slightly improved compared with the MC.

2. Importance Sampling Simulation

The IS technique is widely used for various types of engineering problems, e.g., for estimation of the blocking probability in queueing system [1], [2], [4], [12]–[14], the error rate in communications system [1], [3], [5], etc. See [1] for overview of the application of the IS simulation.

In case that the target event (a blocking in queueing or an error in communications system) is a rare event with small probability of the order less than 10^{-6}, it is impossible to obtain an estimate by the ordinary MC method. The MC method requires considerable amount of computation time. To overcome this difficulty, the underlying probability distribution is modified to generate more samples in the target event. Then the samples obtained by the modified distribution form an unbiased estimate of the true probability. This estimate is called an IS estimate. The modified distribution is called a simulation distribution. If the simulation distribution is appropriately chosen, the variance of the IS estimate can be smaller than that of the estimate by the ordinary MC simulation. The simulation distribution that yields the IS estimate with the minimum variance is referred to as the optimal simulation distribution. When we apply the IS technique to some simulation problems, it is critical to find the optimal simulation distribution. In [10], we investigated the IS simulation for the sample average of output sequences from an irreducible Markov chain and studied the geometric nature of the optimal simulation distribution.

The application of the IS technique to the simulation of queueing problems has been studied by many authors, but the methods of determining the optimal

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IS simulation distribution are heuristic. In [14], the exhaustive search of some region of the system parameters is done to obtain the optimal simulation distribution. In [4], a so-called mean field annealing algorithm is used to find the optimal simulation distribution. These search algorithms require much computation time. There are some works [1] which take theoretical approach to determine the optimal simulation distribution, but the treated models are rather simple ones, as M/M/1.

Our approach of this paper is theoretical. The target traffic model is an n-state MMPP (Markov modulated Poisson process) which is one of the most important traffic models in ATM networks. The MMPP can represent the bursty nature of cell traffic such as voice or data. Poisson processes, interrupted Poisson processes (IPP) are special cases of MMPP. We apply the large deviations theory, especially the sample path large deviations theory, for Markov chains to obtain the optimal IS simulation distributions.

3. Markov Chain Model of MMPP/D/1

Let us consider an n-state MMPP (Markov modulated Poisson process). Let $S_1, \cdots, S_n$ denote the states of the MMPP, λ_i the mean arrival rate of the Poisson process at the state S_i, $i = 1, \cdots, n$, and r_{ij} the state transition probability from state S_i to S_j, $i, j = 1, \cdots, n$. We will make a discrete time Markov chain model of the MMPP/D/1 queueing system. Define the unit time of the system by the service time for one cell. A time interval of the unit length is called a time slot. In a time slot, one of the states $S_1, \cdots, S_n$ is assigned and the state transition can occur at the beginning of a time slot. The queueing process in a time slot is done in the following order;

1. state transition can occur,
2. the queue length and the state are observed,
3. one cell is served, if any,
4. new cells arrive.

(1)

Let $Q(t)$ denote the queue length observed at t th time slot, and $I_i(t)$ denote the indicator which shows that the state S_i is the observed state at the t th time slot, i.e.,

$$I_i(t) = \begin{cases} 1, & \text{if } S_i \text{ is the state at the } t \text{ th time slot,} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

for $i = 1, \cdots, n$.

Now, let A_i denote the Poisson random variable of rate λ_i that represents the number of arriving cells in a time slot with state S_i. Let T_i denote a random variable that represents a transition from the state S_i, i.e., T_i is a random variable on the set of states $\{S_1, \cdots, S_n\}$ that is defined by $P(T_i = S_j) = r_{ij}$, $i, j = 1, \cdots, n$. Define a $2n$-dimensional i.i.d. random vector W by $W = (A_1, \cdots, A_n, T_1, \cdots, T_n)$.

Let us consider a random vector

$$X(t) = (Q(t), I_1(t), \cdots, I_n(t)),\ t = 0, 1, \cdots \tag{3}$$

and study its probabilistic law. We investigate the difference

$$\begin{aligned} &X(t+1) - X(t) \\ &= (Q(t+1) - Q(t), I_1(t+1) - I_1(t), \cdots, \\ &\qquad I_n(t+1) - I_n(t)),\ t = 0, 1, \cdots. \end{aligned} \tag{4}$$

For simplicity, write $\Delta_t Q \equiv Q(t+1) - Q(t)$, $\Delta_t I_i \equiv I_i(t+1) - I_i(t)$, $i = 1, \cdots, n$. Since $I_i(t)$ is the indicator function of the state of the MMPP at the t th time slot, we can see that the number of arriving cells in the t th time slot is represented by $\sum_{i=1}^{n} I_i(t) A_i$. By the order of the queueing process shown in (1), we have the fundamental recursion;

$$Q(t+1) = \max(0, Q(t) - 1) + \sum_{i=1}^{n} I_i(t) A_i. \tag{5}$$

From (5), we have

$$\begin{aligned} \Delta_t Q &\equiv Q(t+1) - Q(t) \\ &= \max(0, Q(t) - 1) + \sum_{i=1}^{n} I_i(t) A_i - Q(t) \\ &= \begin{cases} \sum_{i=1}^{n} I_i(t) A_i - 1, & \text{for } Q(t) > 0, \\ \sum_{i=1}^{n} I_i(t) A_i, & \text{for } Q(t) = 0. \end{cases} \end{aligned} \tag{6}$$

Next, we define a function δ_i, $i = 1, \cdots, n$, on the set of the states $\{S_1, \cdots, S_n\}$ by

$$\delta_i(S_j) = \delta_{ij} \ \text{(Kronecker's delta).} \tag{7}$$

If S_i is the state at the t th time slot, then the state at the $(t+1)$-st time slot is determined by the random variable T_i. Thus, we have

$$I_i(t+1) = \sum_{j=1}^{n} I_j(t) \delta_i(T_j),\ i = 1, \cdots, n. \tag{8}$$

From (8), we have

$$\begin{aligned} \Delta_t I_i &\equiv I_i(t+1) - I_i(t) \\ &= \sum_{j=1}^{n} I_j(t) \delta_i(T_j) - I_i(t) \\ &= \sum_{j=1}^{n} I_j(t)(\delta_i(T_j) - \delta_{ij}),\ i = 1, \cdots, n. \end{aligned} \tag{9}$$

From (6), (9), we see that $\Delta_t Q, \Delta_t I_i$, $i = 1, \cdots, n$

are deterministic functions of $X(t) = (Q(t), I_1(t), \cdots, I_n(t))$ and $W = (A_1, \cdots, A_n, T_1, \cdots, T_n)$.

Write

$$\Delta(X(t), W) = (\Delta_t Q, \Delta_t I_1, \cdots, \Delta_t I_n). \quad (10)$$

Δ defines a deterministic function from $\mathbb{R}^{n+1} \times \mathbb{R}^{2n}$ to $\mathbb{R}^{n+1}$ where $\mathbb{R}$ denotes the set of real numbers. Consequently, we have the following recursive formula of $X(t)$;

$$X(t+1) = X(t) + \Delta(X(t), W),\ t = 0, 1, \cdots. \quad (11)$$

Since W is an i.i.d. random vector, the random process $X(t)$ forms a Markov chain.

In summary, we have

Theorem 1: Given an initial state X_0, the random process $\{X(t) = (Q(t), I_1(t), \cdots, I_n(t))\}_{t=0,1,\cdots}$ forms a Markov chain with the following recursive formula;

$$X(0) = X_0, \quad (12)$$

$$X(t+1) = X(t) + \Delta(X(t), W),\ t = 0, 1, \cdots, \quad (13)$$

where Δ is a deterministic function from $\mathbb{R}^{n+1} \times \mathbb{R}^{2n}$ to $\mathbb{R}^{n+1}$ and represented by

$$\begin{aligned} &\Delta(X(t), W) \\ &= (\Delta_t Q, \Delta_t I_1, \cdots, \Delta_t I_n) \\ &= \begin{cases} \left(\sum_{i=1}^{n} I_i(t)A_i - 1, \sum_{i=1}^{n} I_i(t)(\delta_1(T_i) - \delta_{1i}), \cdots, \sum_{i=1}^{n} I_i(t)(\delta_n(T_i) - \delta_{ni})\right), & \text{for } Q(t) > 0, \\ \left(\sum_{i=1}^{n} I_i(t)A_i, \sum_{i=1}^{n} I_i(t)(\delta_1(T_i) - \delta_{1i}), \cdots, \sum_{i=1}^{n} I_i(t)(\delta_n(T_i) - \delta_{ni})\right), & \text{for } Q(t) = 0. \end{cases} \end{aligned} \quad (14)$$

We here call the $\{X(t)\}_{t=0,1,\cdots}$ the Markov chain model of the MMPP/D/1 queueing system.

For simplicity, let X represent the $X(t)$ at an arbitrary time slot t. Denote by $P(\Delta(X, W)|X)$ the conditional probability of the random variable $\Delta(X, W)$ given X, and $M_X(\theta)$ the moment generating function of $\Delta(X, W)$ given X. The $M_X(\theta)$ is defined by

$$M_X(\theta) \equiv \sum_{\Delta(X,W)} e^{\theta \cdot \Delta(X,W)} P(\Delta(X, W)|X), \quad (15)$$

where $\theta = (\theta_0, \theta_1, \cdots, \theta_n)$ is the $(n+1)$-dimensional parameter and $\cdot$ represents the scalar product of vectors. For $X = (Q, \overbrace{0, \cdots, 0}^{i-1}, 1, \overbrace{0, \cdots, 0}^{n-i})$, let us consider $\Delta(X, W) \equiv (\Delta Q, \Delta I_1, \cdots, \Delta I_n)$, where $\Delta Q, \Delta I_i$ represent $\Delta_t Q, \Delta_t I_i, i = 1, \cdots, n$ at an arbitrary time t, respectively. Recall that $X = (Q, \overbrace{0, \cdots, 0}^{i-1}, 1, \overbrace{0, \cdots, 0}^{n-i})$ means that the queue length is Q and the state is S_i at the present time slot. Since ΔQ represents the difference of queue lengths between the present and the next time slots, the number of arriving cells in the present time slot is

$$\begin{cases} \Delta Q + 1, & \text{for } Q > 0, \\ \Delta Q, & \text{for } Q = 0. \end{cases} \quad (16)$$

Further, since $\overbrace{0, \cdots, 0}^{i-1}, 1, \overbrace{0, \cdots, 0}^{n-i}$ represents that the state of the MMPP at the present time slot is S_i, the next state is determined by T_i. The state transition probability is written as

$$\sum_{j=1}^{n} r_{ij}\delta_j(T_i),\ i = 1, \cdots, n. \quad (17)$$

From (16), (17), for $X = (Q, \overbrace{0, \cdots, 0}^{i-1}, 1, \overbrace{0, \cdots, 0}^{n-i})$, we have

$$P(\Delta(X, W)|X) = \begin{cases} \dfrac{\lambda_i^{\Delta Q+1}}{(\Delta Q+1)!} e^{-\lambda_i} \sum_{j=1}^{n} r_{ij}\delta_j(T_i), & \text{for } Q > 0, \\ \dfrac{\lambda_i^{\Delta Q}}{\Delta Q!} e^{-\lambda_i} \sum_{j=1}^{n} r_{ij}\delta_j(T_i), & \text{for } Q = 0. \end{cases} \quad (18)$$

If $T_i = S_i$, i.e., if the state is unchanged, then $\Delta I_i = 0,\ i = 1, \cdots, n$, otherwise, if $T_i = S_j,\ i \neq j$, then we have $\Delta I_i = -1, \Delta I_j = +1, \Delta I_k = 0, k \neq i, j$. Thus, we have

$$\begin{aligned} M_X(\theta) &= \sum_{\Delta(X,W)} e^{\theta \cdot \Delta(X,W)} P(\Delta(X, W)|X) \\ &= \sum_{\Delta Q, \Delta I_1, \cdots, \Delta I_n} e^{\theta_0 \Delta Q} e^{\theta_1 \Delta I_1} \cdots e^{\theta_n \Delta I_n} P(\Delta(X, W)|X) \\ &= \begin{cases} \sum_{\Delta Q=-1}^{\infty} \sum_{j=1}^{n} e^{\theta_0 \Delta Q} e^{\theta_j - \theta_i} \dfrac{\lambda_i^{\Delta Q+1}}{(\Delta Q+1)!} e^{-\lambda_i} r_{ij}, & \text{for } Q > 0, \\ \sum_{\Delta Q=0}^{\infty} \sum_{j=1}^{n} e^{\theta_0 \Delta Q} e^{\theta_j - \theta_i} \dfrac{\lambda_i^{\Delta Q}}{\Delta Q!} e^{-\lambda_i} r_{ij}, & \text{for } Q = 0. \end{cases} \end{aligned} \quad (19)$$

We can carry out the calculation of the summation in (19) with respect to ΔQ to have

$$M_X(\theta) = \begin{cases} \sum_{j=1}^{n} e^{-\theta_0 + \lambda_i e^{\theta_0} - \lambda_i} r_{ij} e^{\theta_j - \theta_i}, & \text{for } Q > 0, \\ \sum_{j=1}^{n} e^{\lambda_i e^{\theta_0} - \lambda_i} r_{ij} e^{\theta_j - \theta_i}, & \text{for } Q = 0. \end{cases} \quad (20)$$

In summary, we have

Theorem 2: For $X = (Q,\overbrace{0,\cdots,0}^{i-1},1,\overbrace{0,\cdots,0}^{n-i}), i = 1,\cdots,n$, the moment generating function $M_X(\theta)$ of the random variable $\Delta(X,W)$ given X is

$$M_X(\theta) = \begin{cases} \sum_{j=1}^{n} e^{-\theta_0+\lambda_i e^{\theta_0}-\lambda_i} r_{ij} e^{\theta_j-\theta_i}, & \text{for } Q>0, \\ \sum_{j=1}^{n} e^{\lambda_i e^{\theta_0}-\lambda_i} r_{ij} e^{\theta_j-\theta_i}, & \text{for } Q=0. \end{cases} \tag{21}$$

4. Optimal Simulation Distribution

In the previous section, we have shown that the MMPP/D/1 queueing system is represented by a Markov chain model $\{X(t)\}_{t=0,1,\cdots}$, which is dominated by the conditional probability $P(\Delta(X,W)|X)$. In the IS simulation, the underlying probability distribution $P(\Delta(X,W)|X)$ is biased to have a simulation distribution. It is the most important to obtain the simulation distribution that yields the estimate of the minimum variance. Such a simulation distribution is called the optimal simulation distribution. In the following, we will find the optimal simulation distribution of the IS simulation for our problem. The following formalization is due to [1], [2].

Let us consider the following process $X(t;q)$ for $q = 1,2,\cdots,\ t = 0,1,\cdots,;$

$$X(t+1;q) = X(t;q) + \frac{1}{q}\Delta(X(t;q),W). \tag{22}$$

For $\tau = \frac{1}{q}t+\alpha\frac{1}{q}$ with $0 \leqq \alpha < 1$, i.e., $\frac{1}{q}t \leqq \tau < \frac{1}{q}(t+1)$, we define a new process $X_q(\tau)$ by

$$X_q(\tau) \equiv X(t;q) + \alpha(X(t+1;q) - X(t;q)), \tag{23}$$

that is the linear interpolation of $X(t;q)$ and $X(t+1;q)$ (see [1]). Note that t is a discrete parameter and τ is a continuous one. Denote by P_q the associated probability distribution of the process $X_q(\tau)$ (see [2]). $X_q(\tau)$ can be considered as a continuous approximation of the original process $X(t)$. That the first component $Q(t)$ of the original process $X(t)$ exceeds q is equivalent to that the first component of $X_q(\tau)$ exceeds 1.

Now, let us consider another input n-state MMPP and denote its corresponding queueing process by $\widetilde{X}(t)$ and the random variable by $\widetilde{W}$. Denote by $\widetilde{P}(\Delta(\widetilde{X},\widetilde{W})|\widetilde{X})$ the conditional probability that dominates the process $\widetilde{X}(t)$. From $\widetilde{X}(t)$, we can construct a piecewise linear process $\widetilde{X}_q(\tau)$ by the same procedure as (23) whose associate probability is denoted by $\widetilde{P}_q$. For a sample path $\widetilde{X}_q(\tau),\ \tau \geqq 0$, let C_q be the set of paths $\widetilde{X}_q(\tau)$ whose first component starts at 0 and exceeds 1 at some time $\tau_0 \geqq 0$ before returning to 0. Then, the IS estimate $\eta_{\widetilde{P}_q}(C_q)$ for the survivor function $P(Q>q)$ of the MMPP/D/1 queueing is obtained as follows. Let $s_1,\cdots,s_m$ be independent sample paths generated by $\widetilde{P}_q$. The IS estimate $\eta_{\widetilde{P}_q}(C_q)$ is given by

$$\eta_{\widetilde{P}_q}(C_q) = \frac{1}{m}\sum_{j=1}^{m} I_{C_q}(s_j)\frac{dP_q}{d\widetilde{P}_q}(s_j), \tag{24}$$

where I_{C_q} denotes the indicator function of the set C_q. The variance of $\eta_{\widetilde{P}_q}(C_q)$ is

$$Var[\eta_{\widetilde{P}_q}(C_q)] = \int_{C_q}\left(\frac{dP_q}{d\widetilde{P}_q}(s) - E[\eta_{\widetilde{P}_q}(C_q)]\right)^2 d\widetilde{P}_q(s). \tag{25}$$

We say that the simulation distribution $\widetilde{P}$ is the *optimal* if $\widetilde{P}$ gives the minimum of $\lim_{q\to\infty} Var[\eta_{\widetilde{P}_q}(C_q)]$.

A *twisted Markov chain* $P_\theta(\Delta(X,W)|X)$ is a Markov chain of the form

$$P_\theta(\Delta(X,W)|X) = \frac{e^{\theta\cdot\Delta(X,W)}}{M_X(\theta)}P(\Delta(X,W)|X), \tag{26}$$

where $P(\Delta(X,W)|X)$ is the underlying probability distribution. We search the optimal simulation distribution P^* in the class of twisted Markov chains. In fact, the following theorem holds.

Theorem 3: (See [2]) Among all the twisted Markov chains, $P_{\theta^*}(\Delta(X,W)|X)$ with $M_X(\theta^*) = 1$ gives the optimal simulation distribution.

From Theorem 3, we know that the optimal simulation distribution P_{θ^*} is

$$P_{\theta^*}(\Delta(X,W)|X) = e^{\theta^*\cdot\Delta(X,W)}P(\Delta(X,W)|X), \tag{27}$$

where θ^* is the solution of the equation $M_X(\theta) = 1$. We need to solve the equation $M_X(\theta) = 1$ to have the optimal IS simulation distribution. From Theorem 2, for $Q > 0$, the equation $M_X(\theta) = 1$ is equivalent to

$$\sum_{j=1}^{n} e^{-\theta_0+\lambda_i e^{\theta_0}-\lambda_i} r_{ij} e^{\theta_j} = e^{\theta_i},\ i = 1,\cdots,n. \tag{28}$$

Define a matrix $R = (R_{ij})_{i,j=1,\cdots,n}$ by

$$R_{ij} = e^{-\theta_0+\lambda_i e^{\theta_0}-\lambda_i} r_{ij},\ i,j = 1,\cdots,n, \tag{29}$$

and a vector $z = (z_i)_{i=1,\cdots,n}$ by

$$z_i = e^{\theta_i},\ i = 1,\cdots,n. \tag{30}$$

Then, (28) is written as

$$Rz = z. \tag{31}$$

From (31), we see that 1 is an eigen value of R, and z is an eigen vector associated to 1. The matrix R contains only θ_0 among the components of θ, hence the solution θ_0^* is determined by the characteristic equation

$$det(R - E_n) = 0, \tag{32}$$

where E_n denotes the $n \times n$ unit matrix. The other components $\theta_1^*, \cdots, \theta_n^*$ are obtained by the eigen vector z up to a constant multiple of $e^{\theta_i^*}$, $i = 1, \cdots, n$.

Let λ_i^*, r_{ij}^*, $i, j = 1, \cdots, n$ denote the parameters of the optimal MMPP. From (18), (27), by assuming $T_i = S_j$, we put

$$\frac{\lambda_i^{*\Delta Q+1}}{(\Delta Q+1)!} e^{-\lambda_i^*} r_{ij}^* = e^{\theta_0^* \Delta Q} e^{\theta_j^* - \theta_i^*} \frac{\lambda_i^{\Delta Q+1}}{(\Delta Q+1)!} e^{-\lambda_i} r_{ij} \tag{33}$$

for $i, j = 1, \cdots, n$. By comparing the both sides of (33), we see that

$$\lambda_i^* = \lambda_i e^{\theta_0^*}, \ i = 1, \cdots, n, \tag{34}$$

$$r_{ij}^* = r_{ij} e^{\lambda_i^* - \lambda_i - \theta_0^* + \theta_j^* - \theta_i^*}, \ i, j = 1, \cdots, n, \tag{35}$$

satisfy (33).

For $Q = 0$, we can obtain the optimal MMPP in the same way.

In summary, we have

Theorem 4: For an n-state MMPP with parameters λ_i, r_{ij}, $i, j = 1, \cdots, n$, the optimal IS simulation distribution of MMPP/D/1 queueing is given by the MMPP with the following parameters;

$$\lambda_i^* = \lambda_i e^{\theta_0^*}, \tag{36}$$

$$r_{ij}^* = \begin{cases} r_{ij} e^{\lambda_i^* - \lambda_i - \theta_0^* + \theta_j^* - \theta_i^*}, & \text{for } Q > 0, \\ r_{ij} e^{\lambda_i^* - \lambda_i + \theta_j^* - \theta_i^*}, & \text{for } Q = 0, \end{cases} \tag{37}$$

for $i, j = 1, \cdots, n$, where $\theta^* = (\theta_0^*, \theta_1^*, \cdots, \theta_n^*)$ is the solution of the equation $M_X(\theta) = 1$ with $X = (Q, I_1, \cdots, I_n)$.

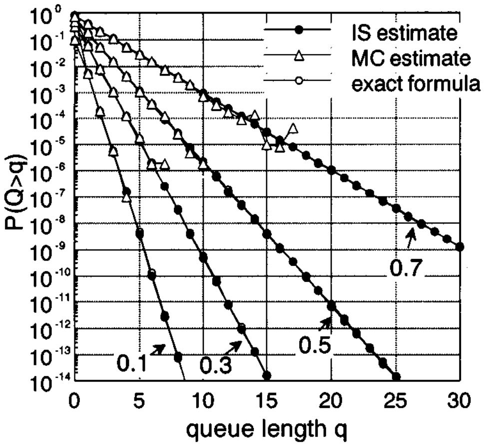

Fig. 1 IS and MC estimates for the survivor function $P(Q > q)$ of M/D/1 with arrival rates 0.1, 0.3, 0.5, 0.7.

5. Numerical Results

We carry out IS simulation with the optimal simulation distribution obtained from Theorem 4 using the notion of regenerative cycle and dynamic importance sampling technique [4], [11]. We will compare IS estimates of $P(Q > q)$ with MC estimates in the following cases.

(1) M/D/1 with the mean arrival rates λ = 0.1, 0.3, 0.5, 0.7 (see Fig. 1).

(2) 2-state MMPP/D/1 with parameters in Table 1 (see Fig. 2).

Table 1 Parameters of the 2-state MMPP's in Fig. 2.

	λ_1	λ_2	r_{12}	r_{21}	λ
Case 1	0.3	0.05	0.6	0.3	0.13
Case 2	0.9	0.1	0.7	0.2	0.27
Case 3	0.8	0.5	0.1	0.4	0.74

(λ denotes the mean arrival rate)

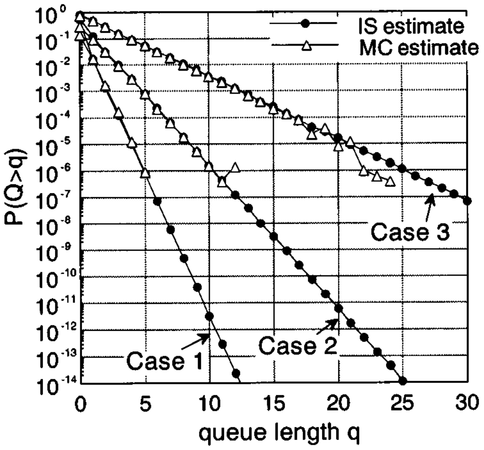

Fig. 2 IS and MC estimates for the survivor function $P(Q > q)$ of 2-state MMPP/D/1.

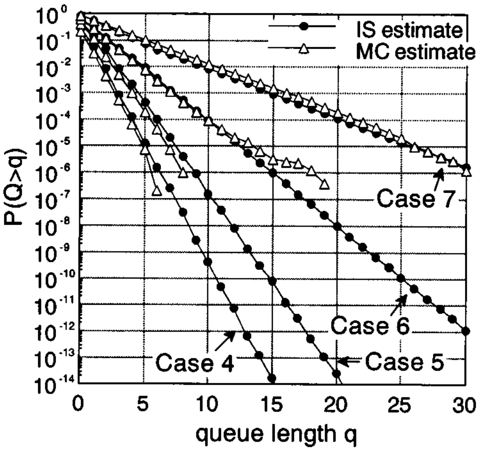

Fig. 3 IS and MC estimates for the survivor function $P(Q > q)$ of 3-state MMPP/D/1.

Table 2 Parameters of the 3-state MMPP's in Fig. 3.

	λ_1	λ_2	λ_3	r_{12}	r_{13}	r_{21}	r_{23}	r_{31}	r_{32}	λ
Case 4	0.1	0.2	0.4	0.4	0.3	0.6	0.1	0.2	0.4	0.2
Case 5	0.3	0.5	0.4	0.4	0.1	0.4	0.2	0.2	0.2	0.4
Case 6	0.8	0.6	0.4	0.1	0.4	0.5	0.2	0.1	0.3	0.6
Case 7	0.9	0.8	0.6	0.4	0.2	0.3	0.2	0.5	0.4	0.8

(λ denotes the mean arrival rate)

Table 3 The length of 95% confidence intervals of IS and MC simulations.

simulation conditions	$P(Q > q)$	confidence interval IS	confidence interval ordinary MC
M/D/1, $\lambda = 0.1$, $q = 7$	1.6×10^{-12}	3.1×10^{-13}	1.5×10^{-12}
M/D/1, $\lambda = 0.7$, $q = 39$	1.8×10^{-12}	2.2×10^{-13}	1.5×10^{-12}
2-state MMPP/D/1, Case 1, $q = 10$	3.3×10^{-12}	4.2×10^{-13}	1.5×10^{-12}
2-state MMPP/D/1, Case 3, $q = 52$	2.6×10^{-12}	3.3×10^{-13}	1.5×10^{-12}

Table 4 The comparison of the computation time of IS and MC in M/D/1.

	average of 10 samples	variance of 10 samples	computation time /sample
IS	8.2×10^{-6}	1.4×10^{-2}	0.01[sec]
MC	7.9×10^{-6}	1.5×10^{-2}	67.7[sec]

(3) 3-state MMPP/D/1 with parameters in Table 2 (see Fig. 3).

We see from Figs. 1–3 that by the ordinary MC simulation we can obtain estimates of $P(Q > q)$ only up to about 10^{-6}. Meanwhile, by the IS simulation we can get stable estimates of all values of $P(Q > q)$. According to our extensive results, we could have values of $P(Q > q) \approx 10^{-200}$.

In the range around $P(Q > q) \approx 10^{-12}$ of M/D/1, the number of time slots in an IS simulation run is 10^6. Since the MC method cannot yield an estimate of a probability 10^{-12}, we assume that it takes about 10^{13} time slots in a MC simulation run. Thus, the ratio of the computation time of the IS simulation to that of the MC simulation is about 10^{-7}. We show in Table 3 the 95% confidence intervals of IS and MC estimates. For MC, we assume conservative values [6] of the confidence intervals in Table 3. We can see that the IS method can reduce the simulation time by improving slightly the 95% confidence intervals of the estimates.

We finally show the comparison of the computation time between IS and MC methods in the case of M/D/1, such that the mean arrival rate $\lambda = 0.5$, $q = 9$, hence the exact value of $P(Q > 9) = 8.1 \times 10^{-6}$. In one simulation run, IS consumed about 10^3 time slots and MC 10^7 time slots. We repeated IS and MC simulation runs 10 times, respectively, and then obtained sample averages and sample variances for $P(Q > 9)$. The obtained sample averages by IS and MC are almost the same, so this is a fair comparison. We have Table 4.

The ratio of the number of consumed time slots of MC to that of IS is about 10^4, but the ratio of the computation time is about $67.7/0.01 = 6.77 \times 10^3$, that is less than 10^4. This is because the IS method requires some additional computational overhead to the MC method.

6. Conclusion

We investigated an importance sampling (IS) simulation for the survivor function $P(Q > q)$ of an n-state MMPP/D/1 queueing systems.

We provided a discrete time Markov chain model of MMPP/D/1 and represented it by a recursive formula. Based on the Markov chain model, we determined the optimal IS simulation distribution by using large deviations theory. Our approach is theoretical, therefore there is no need to do an exhaustive search of the parameter space.

Next, we carried out the IS and Monte Carlo (MC) simulations to compare estimates of $P(Q > q)$ in the cases of M/D/1, 2-state MMPP/D/1, and 3-state MMPP/D/1. We confirmed that the IS method has made the computation time much smaller than the MC method, and the 95% confidence interval slightly improved.

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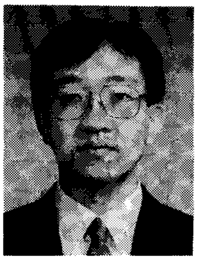

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